Conductance of superconducting-normal hybrid structures

O. Entin-Wohlman,[1,2,*] Y. Imry,[3] and A. Aharony[1,†]
[1]*Department of Physics, Ben Gurion University, Beer Sheva 84105, Israel*
[2]*Albert Einstein Minerva Center for Theoretical Physics, Weizmann Institute of Science, Rehovot 76100, Israel*
[3]*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*
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The dc conductance of normal-superconducting hybrid structures is discussed. It is shown that since the Bogoliubov-DeGennes (BDG) equation does not conserve charge, its application to create a Landauer-type approach for the conductance of the *NSN* system is problematic. We "mend" this deficiency by calculating the conductance from the Kubo formula for a ring configuration, where charge conservation is imposed by the annular geometry. We show that the presence of a superconductor segment within an otherwise normal metal may *reduce* the overall conductance of the composite structure. This reduction enhances the tendency of the *NS* composite to become insulating.

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I. INTRODUCTION

In a seminal paper published a long time ago, Blonder-Tinkham-Klapwijk[1] (BTK) calculated the conductance of a normal (N)-superconducting (S) interface as a function of the interface transparency. In particular they showed that at zero temperature ($T=0$) and for electrons at the Fermi energy that conductance is given by

$$G_{NS}=\frac{2e^2}{h}(1-|S^{ee}|^2+|S^{he}|^2), \tag{1}$$

where $|S^{ee}|^2$ is the "normal" reflection coefficient of the interface, while $|S^{he}|^2$ is the reflection coefficient for the Andreev processes,[2] which reflect electronlike excitations as holelike ones. For simplicity, both the normal and the superconductor regions were taken to be free of any impurity scattering (except at the interface). Then, when the interface is perfectly transparent, i.e., $|S^{ee}|^2=0$ and $|S^{he}|^2=1$, the value of G_{NS} is twice that of the normal quantum limit of the conductance. However, when a large enough barrier exists at the interface between the superconducting and the normal regions, G_{NS} becomes much smaller than the conductance obtained when the superconductor is made normal. This is the simplest example where superconductivity in a part of a system *reduces* its overall conductance. (The more complicated many channel, disordered case is not addressed in this paper.) In this paper we consider the more subtle *NSN* combination, and demonstrate a similar effect: with a large enough barrier at even one of the *NS* interfaces, the appearance of superconductivity in the S region reduces the overall conductance.

Some of the motivation for the present work comes from our wish to understand why the mixed *NS* bulk composite structure is often insulating at $T=0$ and the superconducting phase of a thin film goes over (with increasing disorder or decreasing film thickness) directly into the insulating rather than to the normal-conducting phase.[3,4] It is interesting that often the activation energy for the conductance of the insulating phase is given by the superconductor gap of the superconducting component.[4] Thus, the superconducting component plays the role of an additional barrier between the normal segments. The charge-vortex duality[5] for a system of charged bosons of course explains the insulating phase as dual to the superconducting one. Vortex localization yields zero resistance and charge localization yields zero conductance at $T=0$. Our purpose is to provide a heuristic understanding of how the charge localization is established. Reducing the small-scale conductance of the system pushes it toward the insulating state. The *NSN* system is the simplest microscopic element of the *NS* network. We find that it already presents nontrivial theoretical questions having to do with a deficiency of the Bogoliubov-DeGennes (BDG) formulation.

Blonder *et al.*[1] employed the BDG equation for the quasiparticle excitations in the superconducting region, assuming that the energy gap Δ which vanishes in the normal part does not vary spatially in the superconductor. Since in most situations the superconducting coherence length ξ is much larger than the Fermi wavelength and much smaller than the length of the S region, this assumption seems quite harmless. Consequently, it is widely accepted that the use of the BDG equation, without attempting to compute the (complex, in general) superconducting order-parameter self-consistently, is valid for many hybrid structures.

A particularly important issue emphasized by BTK concerns the conversion of the normal current into a supercurrent at the *NS* interface, ensuring charge-current conservation over the entire structure (see also Ref. 6). Blonder *et al.*[1] showed that the normal charge current (which is distinct from the quasiparticle-number current) entering the superconductor decays, but concomitantly a supercurrent grows up gradually, until very far inside S the normal current disappears completely, and the entire charge is carried away by Cooper pairs. One might wonder what happens to this scenario when the superconductor has a finite width and is not infinite as in the BTK case. It turns out that within the BDG formulation the supercurrent in the S region does not properly convert back to normal current in the second N region. We are not aware of a way to correct for this deficiency (leaving aside the possibility mentioned above to compute the order-parameter self-consistently). In this paper we will circumvent this problem by using a particular geometry.

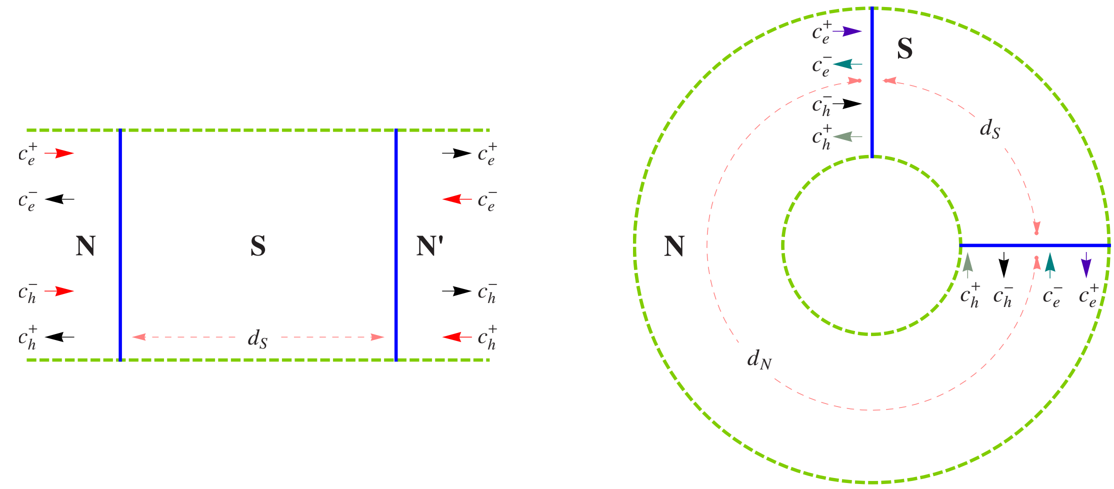

FIG. 1. (Color online) Left panel: the "open" *NSN* structure, where the perpendicular lines represent potential barriers (the explicit model treated in Sec. IV assumes that the right interface is clean). Right panel: the corresponding ring *NSN* system. The incoming and outgoing amplitudes of the electronlike and the holelike waves on both sides of the superconductor are marked by arrows.

Not surprisingly, transport through hybrid *NS* structures has been addressed before, even prior[7] to the publication of the paper by BTK. Indeed, as Eq. (1) bears a strong resemblance to the Landauer formula[8] for coherent transport, several modern treatments (representative examples are to be found in Refs. 9–12) employ scattering theory within the Landauer picture in an attempt to extend the BTK result (1) to more than a single *NS* interface. The scattering formalism is also used to study the current fluctuations in mesoscopic systems with Andreev reflections.[11,12] However, there is a vicious caveat in this approach: the BDG equation, while conserving the number of quasiparticles, does not conserve charge. The BDG formulation follows, in this respect, the similar deficiency of the BCS approach upon which the BDG equation is based. This does not cause any harm when the formulation is used for a bulk superconductor, as had been done by BTK (see also Ref. 13), or when the system has the shape of a closed ring, where the boundary conditions enforce current conservation.[14,15] However, when the size of the superconducting segment is finite, the nonconservation of the charge leads to current vs chemical-potential-difference relations (within the linear-response regime) that depend on the chemical potential of the superconductor, as opposed to the situation in the normal phase.[9,10] Indeed, the conductances of various hybrid structures have been determined within scattering theory (when the superconducting is "floating") by fixing the chemical potential on *S* such that current is conserved.[9,10] One would have assumed naively that this procedure cures the problem mentioned above. It turns out, however, that it does not; as discussed by Anantram and Datta[11] (see also Ref. 12), applying the very same procedure to the calculation of the current fluctuations (the power spectrum of the noise) violates the Johnson-Nyquist relationships. We show in Sec. V that this "floating-potential" approach also does not produce the doubling of the conductance [see Eq. (1)] when the *NS* interface becomes transparent. It approaches, however, the conductance found by the Kubo formula in the limit where the barrier is strong and its transparency is low.

It follows that in order to use the BDG equation for the calculation of the conductance of a mesoscopic system containing a finite-length superconducting segment, one needs charge-conserving solutions of that equation. Such solutions arise naturally when the *NSN* structure (the left panel in Fig. 1) is closed to form a "ring" (the right panel in Fig. 1). One can use those solutions in the Kubo formula for the conductivity of a large system. This is the approach adopted in this paper.

We address the simplest problem of a one-dimensional *NSN* structure at zero temperature (we do not discuss here the complications arising in the multichannel case including disorder). After discussing the conductance between the two *N* segments (left panel in Fig. 1) and dealing with the nontrivial problem arising from the above-mentioned deficiency of the BDG picture, we find that the *NSN* conductance can indeed *decrease* when, for example, the superconducting component becomes longer. We are not able to treat here the behavior of the two-dimensional or the three-dimensional *NSN* arrays. Suffice it to say that when the "small scale" resistance of the elementary building block increases, the tendency for localization at larger scales becomes stronger.

II. DIFFICULTIES WITH THE LANDAUER-TYPE FORMULATION FOR THE *NSN* CONDUCTANCE

The subtleties involved in producing a consistent Landauer-type formula for the conductance of an *NSN* structure within the BDG formulation are best explained by considering the simplest single-mode, two-terminal configuration at zero temperature. In other words, we assume that there is no scattering in the system except for the potential barriers at the interfaces, such that the transverse channel modes are not mixed (and their indices can be omitted).

We start with the purely normal case, as shown in the left panel of Fig. 1, except that the *S* section is replaced by a normal one (for example, by letting its gap approach zero). In this case there is no need to treat electrons and holes concomitantly, as there are no Andreev processes. It is enough to use only electrons (or only holes). We denote the reflection probability for an electron coming from the left by $\mathcal{R}$, and the one for an electron coming from the right by $\mathcal{R}'$. Likewise, the transmission probability from the left to the right is denoted by $\mathcal{T}$, and the one from the right to the left is $\mathcal{T}'$. Unitarity (particle conservation, which is also charge conservation in this case) implies

$$\mathcal{R}+\mathcal{T}=\mathcal{R}'+\mathcal{T}'=1. \tag{2}$$

Time-reversal symmetry implies further that $\mathcal{T}=\mathcal{T}'$, and hence $\mathcal{R}=\mathcal{R}'$. Next we assign to the left conductor a chemical potential μ_L and to the right one a chemical potential μ_R. In the linear-response regime, $\mu_L-\mu_R\to 0$. The middle conductor, of a finite length, is kept floating, (i.e., it is not connected to any reservoir), and will acquire a chemical potential μ_n. Clearly, the right-going currents to the left of the middle segment, I_L, and to its right, I_R, are given by[9]

$$I_L=\frac{2e}{h}[(1-\mathcal{R})(\mu_L-\mu_n)-\mathcal{T}'(\mu_R-\mu_n)],$$

$$I_R=\frac{2e}{h}[-(1-\mathcal{R}')(\mu_R-\mu_n)+\mathcal{T}(\mu_L-\mu_n)]. \tag{3}$$

From the unitarity condition (2) and time-reversal symmetry, it follows that $I_L=I_R\equiv I=(2e/h)\mathcal{T}(\mu_L-\mu_R)$, *independently* of the value of μ_n (which, in fact, drops out of the two equations). The well-known Landauer formula for the conductance,

$$G=(2e^2/h)\mathcal{T}, \tag{4}$$

is immediately obtained, and is independent of μ_n, as it should be.[16]

When the middle section is a superconductor, further Andreev-type processes become possible. An electron can be reflected or transmitted as a hole, and *vice versa.* For an electron incident from the left, the probabilities for the Andreev reflection and transmission processes are denoted $\mathcal{R}_A$ and $\mathcal{T}_A$, respectively. The corresponding quantities for an electron coming from the right are $\mathcal{R}'_A$ and $\mathcal{T}'_A$. The unitary condition (2), which reflects the conservation of quasiparticle *number*, is now replaced by

$$\mathcal{R}+\mathcal{T}+\mathcal{R}_A+\mathcal{T}_A=\mathcal{R}'+\mathcal{T}'+\mathcal{R}'_A+\mathcal{T}'_A=1. \tag{5}$$

However, the *charge* conservation condition [cf. Eq. (2)] now reads[17]

$$\mathcal{R}+\mathcal{T}-\mathcal{R}_A-\mathcal{T}_A=\mathcal{R}'+\mathcal{T}'-\mathcal{R}'_A-\mathcal{T}'_A=1. \tag{6}$$

The two conditions, Eqs. (5) and (6), are *not* compatible whenever the Andreev probabilities do not vanish, *except* for a ring geometry (where their consistency is enforced). Moreover, while Eq. (5) always holds for the solutions of the BDG equation, Eq. (6) does not.

The expressions for the currents [see Eqs. (3)] now become (note that the group velocity of the holes is opposite to that of the electrons)

$$I_L=\frac{2e}{h}[(1-\mathcal{R}+\mathcal{R}_A)(\mu_L-\mu_S)-(\mathcal{T}'-\mathcal{T}'_A)(\mu_R-\mu_S)],$$

$$I_R=\frac{2e}{h}[-(1-\mathcal{R}'+\mathcal{T}'_A)(\mu_R-\mu_S)+(\mathcal{T}-\mathcal{T}_A)(\mu_L-\mu_S)], \tag{7}$$

where μ_S is the chemical potential on the superconductor. We note that because charge conservation does not hold [see Eq. (6)], μ_S does not drop out of these equations. Its value is relevant. Equations (7) are of the same form as Eq. (2) of Ref. 9: these authors determine μ_S so that $I_L=I_R$. It is then possible to obtain a conductance from the current-to-voltage ratio, as was done in Ref. 9. We reproduce their result for the *NSN* conductance in Sec. V, see Eq. (57) there, where we show that it does not agree with the conductance calculated from the linear response, Kubo formula (whereas there is an agreement in the normal case). It is also disturbing that the determined value of μ_S is relevant in the approach of Ref. 9. We return to this point in great detail in Sec. V.

The above considerations, in particular, Eqs. (5) and (6), can be put on a more general basis. By imposing the appropriate boundary conditions on the plane-wave solutions of the BDG equation it is possible to derive the scattering matrix, $\mathcal{S}$, of the *NSN* structure. This (4×4) matrix relates the amplitudes of the incoming waves to those of the outgoing ones, (see left panel in Fig. 1)

$$c_{\text{out}}=\mathcal{S}c_{\text{in}}. \tag{8}$$

Here,[13] the incoming amplitudes are

$$c_{\text{in}}=[c_e^+(N),c_e^-(N'),c_h^-(N),c_h^+(N')], \tag{9}$$

and the outgoing ones are

$$c_{\text{out}}=[c_e^-(N),c_e^+(N'),c_h^+(N),c_h^-(N')]. \tag{10}$$

In Eqs. (9) and (10), $c_{e,h}^+(N)$ denotes the amplitude of an electronlike (holelike) excitation with a positive wave vector $k_{e,h}$ incident from the left normal side while $c_{e,h}^-(N)$ refers to the waves having negative wave vectors. Since the BDG equation conserves the number of quasiparticles, the scattering matrix $\mathcal{S}$ is necessarily unitary, and therefore

$$c_{\text{out}}^\dagger c_{\text{out}}=c_{\text{in}}^\dagger c_{\text{in}}. \tag{11}$$

However, conservation of the charge current[1,17] requires [see Eq. (6)]

$$|c_e^+(N)|^2-|c_e^-(N)|^2+|c_h^+(N)|^2-|c_h^-(N)|^2 = |c_e^+(N')|^2-|c_e^-(N')|^2+|c_h^+(N')|^2-|c_h^-(N')|^2. \tag{12}$$

Comparing Eqs. (11) and (12), we see that they imply

$$|c_e^+(N)|^2+|c_e^-(N')|^2=|c_e^+(N')|^2+|c_e^-(N)|^2,$$

$$|c_h^-(N)|^2+|c_h^+(N')|^2=|c_h^+(N)|^2+|c_h^-(N')|^2. \tag{13}$$

Namely, the sum of the amplitudes squared of the incoming electronlike excitations is equal to the sum of the amplitudes squared of the outgoing electronlike excitations, and so is the situation for the holelike ones. These conditions are the same as those that would have been derived from Eqs. (5) and (6), had we required that both conditions should be satisfied together. This always holds for a normal system, in which these two types of quasiparticles are not mixed. However, in a superconductor the Andreev processes mix the holelike with the electronlike excitations, thus violating the conditions [Eq. (13)] for general hybrid structures with a finite-size S segment. An exception is the ring geometry. There, (see Fig. 1) the ratios $c^{+}_{e,h}(N)/c^{+}_{e,h}(N')$ are necessarily phase factors, and so are the ratios $c^{-}_{e,h}(N)/c^{-}_{e,h}(N')$. As a result, the plane-wave solutions of the BDG equations for the ring geometry do satisfy both conditions [Eq. (13)], namely, these solutions conserve charge. One may therefore employ the BDG equation for the ring geometry in the context of the Kubo formulation to calculate the conductance of the *NSN* structure.

III. THE KUBO FORMULA FOR A LARGE RING

For an infinite system, the Kubo-type conductivity at frequency ω may be most easily obtained by calculating, using the golden rule, the power absorbed by the system from a classical monochromatic electromagnetic field. We consider for simplicity noninteracting fermions (or Fermi quasiparticles), and focus on the σ_{xx} component of the conductivity,

$$\sigma_{xx}(\omega) = -\frac{\pi e^2}{V}\frac{1}{\omega}\sum_{j,\ell}|\langle j|v_x|\ell\rangle|^2 \times\delta(\epsilon_\ell-\epsilon_j-\hbar\omega)[f(\epsilon_j)-f(\epsilon_\ell)]. \quad (14)$$

Here, $|j\rangle$ and $|\ell\rangle$ are the quasielectron states and $f(\epsilon_j)$ and $f(\epsilon_\ell)$ are their populations. In Eq. (14), V is the volume of the system, to be sent to infinity at the end of the calculation, at which stage the summations over the states are replaced by integrations with the densities of states. The x component of the velocity operator is denoted v_x. For the case of a normal (i.e., nonsuperconducting) scatterer *with infinite leads*, the equivalence of the Kubo and the two-terminal Landauer approaches has been established in Refs. 18 and 19.

The assumption of an infinite system is crucial in order to have a continuum of states. An *isolated* finite system with a truly discrete spectrum does *not* in fact absorb energy from the monochromatic field. In order to obtain a finite conductivity for a finite large system, it has to be (and it is, in most real situations) coupled to a very large heat bath, for example, to an assembly of thermal phonons. This enables energy to be transferred from the electromagnetic field into the bath via the small electronic system. For a weak enough interaction with the bath, one may say that the discrete levels of the system have acquired finite widths, η_j. It then makes sense to write down Eq. (14) with the levels having a finite width (or with an imaginary part to the frequency ω, which will amount to a nonmonochromatic driving field). This procedure has been discussed, including the dc limit (Re $\omega\to 0$), by Thouless and Kirkpatrick,[20] following Czycholl and Kramer,[21] and used for example in Ref. 22, see also Ref. 23. It is postulated, and can be demonstrated in typical cases, that once the η_j's are larger than the level spacing near the Fermi energy, but much smaller than all other relevant energy scales in the problem, this procedure yields the physically relevant low-frequency conductance of the system.

It hence follows that the $\omega\to 0$ conductance is obtained upon transforming the summations in Eq. (14) into energy integrations. This allows one to approximate $f(\epsilon_\ell+\hbar\omega)-f(\epsilon_\ell)\simeq\hbar\omega f'(\epsilon_\ell)$. Focusing on our one-dimensional configuration, we take x along the ring circumference and replace the volume of the system by its length, d. Since the (one-dimensional) conductance is related to the conductivity by $G\equiv\sigma/d$, we recover the Kubo-Greenwood-type formula at low frequencies,

$$G=\frac{e^2h}{2}\nu^2\sum_{\rm deg}|\langle|v|\rangle|^2, \quad (15)$$

where the sum is over the (almost) degenerate initial states at the Fermi energy, and over the (almost) degenerate final states, which belong to energies within the narrow range $\hbar\omega(\to 0)$ above those initial states. One might also say that the cancellation of the frequency [see Eq. (14)] is caused by the fact that the initial state was, at $T=0$, within $\hbar\omega$ of the Fermi level. In Eq. (15), the matrix element squared of the velocity was replaced by its typical value in the small relevant energy window around the Fermi energy. The double sum of Eq. (14) gave rise to two factors of the single-particle density of states (per unit energy, per unit length, and per spin), ν,

$$\nu = 1/(hv_F), \quad (16)$$

in the one-dimensional system. Comparing Eq. (15) with the "traditional" Landauer formula (4), we find that in the Kubo approach the total transmission is replaced by the appropriate sum over the velocity matrix elements squared, i.e.,

$$G=\frac{2e^2}{h}\frac{1}{4v_F^2}\sum_{\rm deg}|\langle|v|\rangle|^2. \quad (17)$$

It is instructive to review the way the Kubo formula in its form (17) produces the Landauer result for the usual two-probe geometry. We consider initial left-going scattering states. These are degenerate with the right-going ones. This degeneracy gives a factor of 2 in the final result, to which the spin degeneracy adds another factor of 2. Each such state will have a matrix element, $(1-\mathcal{R}+\mathcal{T})/2=\mathcal{T}$, with the appropriate final left-going scattering state, and rt with the final right-going scattering state (r and t are the reflection and transmission amplitudes, respectively). Adding the absolute values squared together yields $\mathcal{T}$ (note the cancellation of the $\mathcal{T}^2$ term). Introducing the above degeneracy factors gives $\Sigma_{\rm deg}|\langle|v|\rangle|^2=4v_F^2\mathcal{T}$, and thus reproduces the Landauer result, Eq. (4) above.

For the ring geometry the states are stationary and normalizable. Taking as a representative example a normal ring with a single delta-function potential, one finds that the ratio of the amplitudes of the clockwise-moving wave and counterclockwise-moving one is a phase factor, $\exp[i\phi_e]$, where on the Fermi energy $\exp[i\phi_e]=-1$ or $(1-i\zeta)/(1+i\zeta)$. Here ζ is the strength of the delta function potential, with the corresponding transmission $\mathcal{T}=1/(1+\zeta^2)$. The velocity matrix elements are then $v_{\rm F}/(1\pm i\zeta)$, and thus together with the spin degeneracy reproduce the Landauer formula (4). We give more details in Sec. IV, which is devoted to the evaluation of the states and the current matrix elements for an *NS* ring, and the case of an entirely normal ring is treated as a limiting case.

IV. THE VELOCITY MATRIX ELEMENTS

Here we compute the matrix elements of the velocity operator, which are used in the Kubo formula (17) for the conductance. We follow BTK in assuming that the entire scattering takes place only at the *NS* interfaces, and that the pair-potential Δ is finite and spatially invariant in the superconducting region, and vanishes in the normal one. Since then there is no channel mixing, the problem becomes effectively one dimensional, and the quasiparticles are described by the one-dimensional Bogoliubov-DeGennes equation, (we use in this section units in which $\hbar=1$)

$$\begin{bmatrix} -\frac{1}{2m}\frac{d^2}{dx^2}-E_F & \Delta \\ \Delta & \frac{1}{2m}\frac{d^2}{dx^2}+E_F \end{bmatrix}\Psi(x)=\epsilon\Psi(x). \quad (18)$$

Note that this equation takes into account the two possible spin directions pertaining to a certain energy ϵ (measured from the Fermi level). In N, where $\Delta=0$, the solutions of Eq. (18) are

$$\Psi_e^{\pm}(x)=\begin{bmatrix}1\\0\end{bmatrix}e^{\pm ik_e x}, \quad \Psi_h^{\pm}(x)=\begin{bmatrix}0\\1\end{bmatrix}e^{\pm ik_h x}, \quad (19)$$

with the wave vectors

$$k_{e,h}=\sqrt{2m(E_F\pm\epsilon)}\simeq k_F\pm\epsilon/v_F. \quad (20)$$

In the superconducting segment the solutions are

$$\Psi_e^{\pm}(x)=\begin{bmatrix}\tilde{u}\\\tilde{v}\end{bmatrix}e^{\pm iq_e x}, \quad \Psi_h^{\pm}(x)=\begin{bmatrix}\tilde{v}\\\tilde{u}\end{bmatrix}e^{\pm iq_h x}. \quad (21)$$

Here,

$$q_{e,h}=\sqrt{2m(E_F\pm\Omega)}\simeq k_F\pm\Omega/v_F, \quad (22)$$

where

$$\Omega=\sqrt{\epsilon^2-\Delta^2}, \quad \epsilon\geq\Delta,$$

$$\Omega=i\sqrt{\Delta^2-\epsilon^2}, \quad \epsilon\leq\Delta, \quad (23)$$

and

$$\tilde{u}^2=\frac{\epsilon}{\Omega}\left(\frac{1}{2}+\frac{\Omega}{2\epsilon}\right), \quad \tilde{v}^2=\frac{\epsilon}{\Omega}\left(\frac{1}{2}-\frac{\Omega}{2\epsilon}\right). \quad (24)$$

The factor $[\epsilon/\Omega]^{1/2}$ compensates for the different group velocity of the quasiparticles in the superconductor $[\partial\epsilon/\partial q=(q/m)(\Omega/\epsilon)]$, and it multiplies the usual coherence factors u and v, $\tilde{u}=[\epsilon/\Omega]^{1/2}u$ and $\tilde{v}=[\epsilon/\Omega]^{1/2}v$. The amplitudes $c_{e,h}^{\pm}(N)$ (see Fig. 1) are the coefficients of the waves $\exp[\pm ik_{e,h}x]$. Analogous amplitudes are defined for the waves in the superconducting segment.

For simplicity, we assume that the (left) *NS* interface at $x=0$ (see Fig. 1) is represented by a delta-function potential, $\lambda\delta(x)$, of strength $\zeta=\lambda/v_F$. Then the boundary conditions[17] are the continuity of the wave functions and the discontinuity (of magnitude ζ) of their derivatives, leading to the following relations among the amplitudes of the N region and those of the S one,

$$\begin{bmatrix}c_e^+(S)\\c_e^-(S)\\c_h^+(S)\\c_h^-(S)\end{bmatrix}=\begin{bmatrix}\tilde{u}&0&-\tilde{v}&0\\0&\tilde{u}&0&-\tilde{v}\\-\tilde{v}&0&\tilde{u}&0\\0&-\tilde{v}&0&\tilde{u}\end{bmatrix}\begin{bmatrix}1-i\zeta&-i\zeta&0&0\\i\zeta&1+i\zeta&0&0\\0&0&1-i\zeta&-i\zeta\\0&0&i\zeta&1+i\zeta\end{bmatrix}\begin{bmatrix}c_e^+(N)\\c_e^-(N)\\c_h^+(N)\\c_h^-(N)\end{bmatrix}. \quad (25)$$

The other *NS* interface, located at $x=d_S$, is assumed to be perfectly transparent, and then the boundary conditions are the continuity of the wave functions and their derivatives. When the system has the shape of a ring, in which the length of the normal segment is d_N, these boundary conditions are

$$\begin{bmatrix}e^{-ik_Fd_N}\gamma_N^{-1}c_e^+(N)\\e^{ik_Fd_N}\gamma_Nc_e^-(N)\\e^{-ik_Fd_N}\gamma_Nc_h^+(N)\\e^{ik_Fd_N}\gamma_N^{-1}c_h^-(N)\end{bmatrix}=\begin{bmatrix}e^{ik_Fd_S}\tilde{u}\gamma_S&0&e^{ik_Fd_S}\gamma_S^{-1}\tilde{v}&0\\0&e^{-ik_Fd_S}\tilde{u}\gamma_S^{-1}&0&e^{-ik_Fd_S}\gamma_S\tilde{v}\\e^{ik_Fd_S}\tilde{v}\gamma_S&0&e^{ik_Fd_S}\gamma_S^{-1}\tilde{u}&0\\0&e^{-ik_Fd_S}\tilde{v}\gamma_S^{-1}&0&e^{-ik_Fd_S}\gamma_S\tilde{u}\end{bmatrix}\begin{bmatrix}c_e^+(S)\\c_e^-(S)\\c_h^+(S)\\c_h^-(S)\end{bmatrix}, \quad (26)$$

where

$$\gamma_N=e^{i\epsilon d_N/v_F}, \quad \gamma_S=e^{i\Omega d_S/v_F}. \quad (27)$$

Without loss of generality we may choose $\exp(ik_Fd)=1$, where $d=d_N+d_S$ is the total length of the ring. Then k_F disappears from the boundary conditions.

Upon eliminating the S-region amplitudes, one obtains the equation which determines the allowed eigenenergies of the ring, and the ratios among the amplitudes of the normal region for each such energy,

$$\left(\begin{bmatrix} X & 0 & Y & 0 \\ 0 & X^* & 0 & Y^* \\ Y^* & 0 & X^* & 0 \\ 0 & Y & 0 & X \end{bmatrix} - \begin{bmatrix} 1-i\zeta & -i\zeta & 0 & 0 \\ i\zeta & 1+i\zeta & 0 & 0 \\ 0 & 0 & 1-i\zeta & -i\zeta \\ 0 & 0 & i\zeta & 1+i\zeta \end{bmatrix}\right)\begin{bmatrix} c_e^+(N) \\ c_e^-(N) \\ c_h^+(N) \\ c_h^-(N) \end{bmatrix} = 0. \tag{28}$$

Here,

$$X = \gamma_N^{-1}\left(\cos(\Omega d_S/v_F) - i\frac{\epsilon}{\Omega}\sin(\Omega d_S/v_F)\right),$$

$$Y = 2i\gamma_N \tilde{u}\tilde{v}\sin(\Omega d_S/v_F), \tag{29}$$

such that $|X|^2-|Y|^2=1$ for both $\epsilon \geq \Delta$ and $\epsilon \leq \Delta$. The allowed eigenenergies are given by the vanishing of the determinant of the matrix in Eq. (28). The zeroes of the determinant define the families of possible eigenenergies, which are rather dense when the size of the entire system is large. When $\zeta \neq 0$, the eigenvectors of the matrix (28) are such that the ratios of the clockwise electron (hole) waves to the counterclockwise electron (hole) ones (see Fig. 1) for each of the families of eigenenergies are phase factors,

$$\frac{c_e^-(N)}{c_e^+(N)} = e^{i\phi_e}, \quad \frac{c_h^-(N)}{c_h^+(N)} = e^{i\phi_h}. \tag{30}$$

In other words, at any finite value ζ of the barrier, there is a perfect reflection of the electron and the hole waves (in the ring geometry). On the other hand, the ratios of the hole amplitudes to the electron ones obey

$$\frac{c_h^+(N)}{c_e^+(N)} = P, \quad \frac{c_h^-(N)}{c_e^-(N)} = P^*, \tag{31}$$

such that the phase of P is $(\phi_e - \phi_h)/2$,

$$P = |P|e^{i(\phi_e-\phi_h)/2}. \tag{32}$$

It is illuminating to consider Eq. (28) and its solutions in the limit of very high energies, $\epsilon \gg \Delta$, where the superconducting order parameter Δ becomes irrelevant, and the entire system behaves as if it were normal. Then [see Eqs. (29)] $X=\exp[-i\epsilon d/v_F]$ and $Y=0$, and Eq. (28) separates into two independent blocks, for the electronlike excitations, and for the holelike ones. The four families of eigenenergies are determined by

$$e^{i\epsilon d/v_F} = 1 \ \text{ and } \ \frac{1+i\zeta}{1-i\zeta} \quad \text{for the electron waves},$$

$$e^{i\epsilon d/v_F} = 1 \ \text{ and } \ \frac{1-i\zeta}{1+i\zeta} \quad \text{for the hole waves}, \tag{33}$$

and the corresponding phase ratios for the two sets of states are

$$e^{i\phi_e} = e^{i\phi_h} = -1 \quad \text{and} \quad \frac{1-i\zeta}{1+i\zeta}. \tag{34}$$

(In this limit P, the ratio of the hole amplitude to the electron amplitude, is not defined.) We show below that these are the phase factors $\exp[i\phi_{e,h}]$ which determine the conductance (4) when calculated from the Kubo formula (17).

In the other extreme limit of subgap energies, $\epsilon \ll \Delta$, one approximates[1] [see Eq. (23)]

$$\frac{\Omega}{v_F} \simeq \frac{\Delta}{v_F} \equiv \frac{1}{\xi}, \tag{35}$$

and consequently [see Eqs. (29)]

$$X \simeq \gamma_N^{-1}\cosh(d_S/\xi), \quad Y \simeq -i\gamma_N \sinh(d_S/\xi), \tag{36}$$

where ξ is the coherence length in the superconductor. One then finds a quadratic equation for $\cos(\epsilon d_N/v_F)$. We do not present explicit expressions for the solutions and the amplitude ratios since they are rather cumbersome.

The next step in this calculation is to find the normalization of the wave functions, using

$$\int_0^{d_S} dx|\Psi_S(x)|^2 + \int_{d_S}^{d} dx|\Psi_N(x)|^2 = 1. \tag{37}$$

In the N region

$$|\Psi_N(x)|^2 = |c_e^+(N)|^2 + |c_e^-(N)|^2 + \{[e^{2ik_Fx}c_e^-(N)]^* c_e^+(N) + \text{c.c.}\} + (e \to h). \tag{38}$$

When d_N is large, such that the oscillatory terms (in $k_F d_N$) can be ignored, the contribution of the normal part to the normalization integral becomes

$$\int_{d_S}^{d} dx|\Psi_N(x)|^2 = d_N(|c_e^+(N)|^2 + |c_e^-(N)|^2 + |c_h^+(N)|^2 + |c_h^-(N)|^2). \tag{39}$$

The calculation of the contribution to the normalization coming from the S region is more subtle, since the wave vectors can have an imaginary part [see Eqs. (22) and (23)]. Disregarding terms oscillating with $k_F d_S$, we find

$$\begin{aligned}|\Psi_S(x)|^2 \to (|\tilde{u}|^2 + |\tilde{v}|^2)\{[|c_e^+(S)|^2 + |c_h^-(S)|^2]e^{i(\Omega-\Omega^*)x/v_F} \\ + [|c_e^-(S)|^2 + |c_h^+(S)|^2]e^{i(\Omega^*-\Omega)x/v_F}\} \\ + (\tilde{u}^*\tilde{v} + \tilde{u}\tilde{v}^*)\{e^{i(\Omega+\Omega^*)x/v_F}[(c_e^-(S))^* c_h^-(S) \\ + (c_h^+(S))^* c_e^+(S)] + \text{c.c.}\}.\end{aligned} \tag{40}$$

This rather complicated result reflects the fact (specifically, its second part) that in the superconductor the electron waves are mixed with the hole ones. However, at very large energies, $\epsilon \gg \Delta$, or at very small ones, $\epsilon \ll \Delta$, the mixing term, $(\tilde{u}^*\tilde{v}+\tilde{u}\tilde{v}^*)$, vanishes. In the following, we confine ourselves to these two limits. In the high-energies limit the normalization of either the clockwise waves or the counterclockwise ones is simply $\sqrt{2d}$, where d is the total length. In the low-energies limit we find

$$\int_0^{d_S} dx|\Psi_S(x)|^2 = \frac{\xi}{2}\sinh(d_S/\xi)(|c_e^+(N)|^2 + |c_e^-(N)|^2)(e^{d_S/\xi}|M_a|^2 + e^{-d_S/\xi}|M_b|^2), \quad (41)$$

where

$$M_a = \gamma_N^{-1} - i\gamma_N P,$$

$$M_b = \gamma_N^{-1} + i\gamma_N P, \quad (42)$$

and P is given by Eq. (31).

Having fully determined the wave functions, it remains to compute the matrix elements of the velocity,

$$v_{j\ell} \equiv \langle j|v|\ell\rangle = \frac{1}{2mi}\int_0^d dx\left(\Psi_j^*\frac{d\Psi_\ell}{dx} - \Psi_\ell\frac{d\Psi_j^*}{dx}\right), \quad (43)$$

with the indices j and ℓ enumerating the various eigenfunctions. As in the calculation of the normalization, here again there are contributions from the normal and from the superconducting regions. In each region we discard the oscillatory terms, those which involve $k_F d_N$ or $k_F d_S$.

The contribution of the normal part to the integral in Eq. (43) reads

$$v_{j\ell}^N = \frac{d_N k_F}{m}\{[c_e^+(N)]_j^*[c_e^+(N)]_\ell - [c_e^-(N)]_j^*[c_e^-(N)]_\ell + [c_h^+(N)]_j^*[c_h^+(N)]_\ell - [c_h^-(N)]_j^*[c_h^-(N)]_\ell\}. \quad (44)$$

In the high-energies limit, $\epsilon \gg \Delta$, the contribution of the superconducting segment to the integration is the same as Eq. (44) (with the arguments N replaced by S, and d_N replaced by d_S). The contribution of the S part in the limit of very low energies, $\epsilon \ll \Delta$, is

$$v_{j\ell}^S = \frac{\xi k_F}{2m}\sinh(d_S/\xi)\{[c_e^+(N)]_j^*[c_e^+(N)]_\ell M_{j\ell} - [c_e^-(N)]_j^*[c_e^-(N)]_\ell M_{j\ell}^*\}, \quad (45)$$

where we have denoted [see Eqs. (42)]

$$M_{j\ell} = e^{d_S/\xi}(M_a)_j^*(M_a)_\ell + e^{-d_S/\xi}(M_b)_j^*(M_b)_\ell. \quad (46)$$

It is again useful to examine the limit of high energies, $\epsilon \gg \Delta$, where the entire ring behaves as if it were normal. Then, the electronlike and the holelike waves are separated. The spectrum and the amplitude ratios are given by Eqs. (33) and (34), and the normalization for each species is $\sqrt{2d}$. The matrix elements of the velocity are simply

$$v_{j\ell}^{\text{ele}} = \frac{k_F d}{m}(c_e^+)_j^*(c_e^+)_\ell(1 - e^{i(\phi_e^\ell - \phi_e^j)}), \quad (47)$$

and an analogous result is obtained for the contribution of the hole waves. Obviously, the diagonal ones vanish. The nondiagonal ones give $(k_F/m)/(1 \pm i\zeta)$, and consequently

$$\sum_{\text{deg}}|\langle|v|\rangle|^2 = 4v_F^2\mathcal{T}, \quad \mathcal{T} = \frac{1}{1+\zeta^2}. \quad (48)$$

Note that the nonvanishing matrix elements arise from the phase factor between waves belonging to the same species but moving along opposite directions. Hence, the Kubo formulation for the ring geometry reproduces the Landauer result for the dc conductance.

Another illuminating limit is when ζ vanishes, and both *NS* interfaces (see Fig. 1) are perfectly transparent, in which case the clockwise and the counterclockwise amplitudes are independent. The matrix elements of the velocity, for subgap energies, are (for either the clockwise-moving or the counterclockwise-moving excitations)

$$v_{\ell\ell} = v_F,$$

$$v_{j\ell} = v_{\ell j}^* = v_F\frac{d_N}{d_N + \xi\tanh(d_S/\xi)}\frac{\sinh(d_S/\xi)[\sinh(d_S/\xi) + i]}{\cosh^2(d_S/\xi)}. \quad (49)$$

(It is interesting to note that the off-diagonal matrix elements are coming from the N region alone.) Hence the contribution of both the clockwise waves and the counterclockwise waves is

$$\sum_{\text{deg}}|\langle|v|\rangle|^2 = 4v_F^2\left\{1 + \left[\frac{d_N\tanh(d_S/\xi)}{d_N + \xi\tanh(d_S/\xi)}\right]^2\right\}. \quad (50)$$

Thus, when d_S/ξ tends to zero (namely, in the absence of the superconductor) the result approaches the Landauer formula for a transparent barrier, cf. Eq. (48). On the other hand, when $d_N \geq d_S \gg \xi$, our result (50) tends to the one found by BTK (Ref. 1) (for a clean interface), namely, it is *twice* the value of the quantum conductance.

Unfortunately, the explicit expressions for the velocity matrix elements at low energies for general values of ζ and d_S/ξ are rather complicated. Consequently, we present the results of the calculations only graphically, see Figs. 2 and 3. The figures show the conductance [divided by $2e^2/h$, see Eq. (17)] as a function of the ratio d_S/ξ for various values of the interface transmission, $\mathcal{T}=1/(1+\zeta^2)$, and as a function of that transmission, for various values of the size of the superconductor segment, d_S/ξ.

Figure 2 presents the results for the case $d_N=d_S$, i.e., the segments N and S of the ring are of equal lengths. In the left panel the conductance is plotted as a function of d_S/ξ, for values of $\mathcal{T}$ ranging between 1 (the upmost curve) and 0.06, (the lowest-lying one).[24] The main feature of these curves is the variation in their slope as the *NS* barrier becomes less and less transparent. For $\mathcal{T}=1$ the conductance increases with the length of the superconductor (until it is double that of the normal system in the BTK limit where $d_S/\xi \to \infty$). As

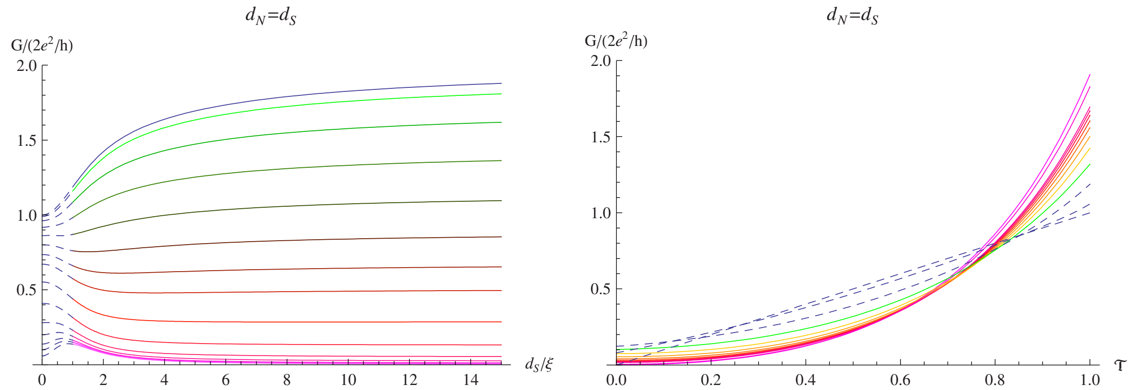

FIG. 2. (Color online) Left panel: conductance vs d_S/ξ for several values of the transmission. Right panel: conductance vs the transmission for several values of d_S/ξ. Here the length of the S region equals that of the N region ($d_N = d_S$). All information pertaining to values of d_S smaller than ξ is presented by dashed curves.

the transparency decreases, the conductance, albeit increasing with d_S/ξ becomes smaller, until at about $\mathcal{T} \simeq 0.8$ it changes its slope and begins *deceasing* as the size of the superconductor is increased. The same characteristic behavior is obtained when the size of the normal part largely exceeds that of the superconductor, as is depicted in the left panel of Fig. 3. The right panels in both Figs. 2 and 3 show the (normalized) conductance as a function of the barrier transparency for various values of the superconducting size, d_S/ξ, ranging between 0.01 (almost a straight line) and 20 (parabolic curve). Here one observes that the conductance is linear in the barrier transmission as long as the superconducting is small enough, and then becomes quadratic in $\mathcal{T}$, for large values of d_S/ξ. It should be noted, however, that the use of the BDG approach for $d_S/\xi \ll 1$ is dubious. For this reason, we have presented all information pertaining to such values by the dotted curves.

V. DISCUSSION

As is described in Secs. I and II, several previous calculations aiming to determine the conductance of hybrid normal-superconducting structures are based on the scattering matrix for the quasiparticles, as derived from the BDG equation.[9–11] Our reservations regarding this procedure are explained in Sec. II. Nonetheless, it is interesting to compare the conductance found from the Kubo formula and the one derived after fixing the chemical potential of the superconductor, as explained in Sec. II. Here we carry out this comparison for the model system of Sec. IV.

The scattering matrix of the NSN junction (see the left panel in Fig. 1) is a function of the energy ϵ. For our purposes here it suffices to derive it for zero energy, i.e., on the Fermi level. This derivation is accomplished by eliminating the amplitudes of the waves within the superconductor, using the boundary conditions (25), and the boundary conditions at the (clean) interface between the superconductor and the second normal layer, denoted N' [note that when $\epsilon = 0$, $\gamma_N = 1$, see Eq. (27)]

$$\begin{bmatrix} c_e^+(N') \\ c_e^-(N') \\ c_h^+(N') \\ c_h^-(N') \end{bmatrix} = \begin{bmatrix} \tilde{u}\gamma_S & 0 & \gamma_S^{-1}\tilde{v} & 0 \\ 0 & \tilde{u}\gamma_S^{-1} & 0 & \gamma_S\tilde{v} \\ \tilde{v}\gamma_S & 0 & \gamma_S^{-1}\tilde{u} & 0 \\ 0 & \tilde{v}\gamma_S^{-1} & 0 & \gamma_S\tilde{u} \end{bmatrix} \begin{bmatrix} c_e^+(S) \\ c_e^-(S) \\ c_h^+(S) \\ c_h^-(S) \end{bmatrix}. \quad (51)$$

As a result, the scattering matrix as defined in Eq. (8) takes the form

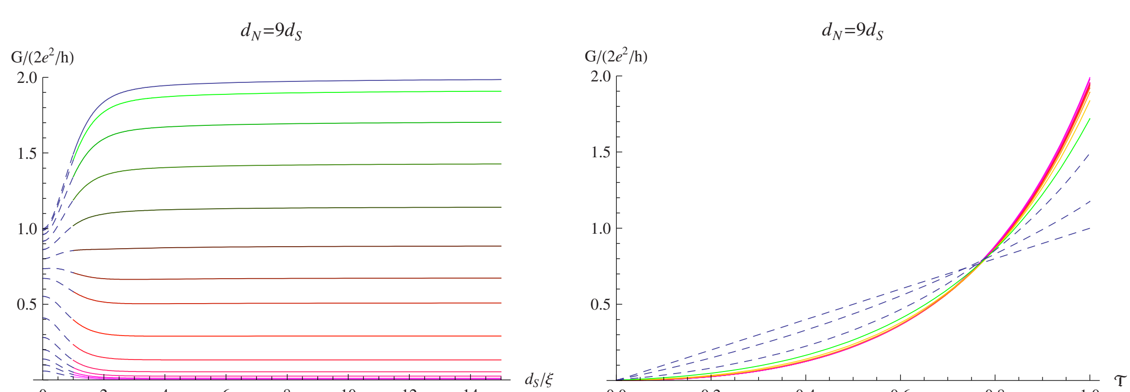

FIG. 3. (Color online) Left panel: conductance vs d_S/ξ for several values of the transmission. Right panel: conductance vs the transmission for several values of d_S/ξ. Here the length of the S region is smaller than the length of the N region ($d_N/d_S = 9$). All information pertaining to values of d_S smaller than ξ is presented by dashed curves.

$$\begin{bmatrix} c_e^-(N) \\ c_e^+(N') \\ c_h^+(N) \\ c_h^-(N') \end{bmatrix} = \frac{1}{D}\begin{bmatrix} -i\zeta(1-i\zeta)(c^2+s^2) & (1-i\zeta)c & isc & -\zeta s \\ c(1-i\zeta) & -i\zeta(1-i\zeta) & \zeta s & isc(1+2\zeta^2) \\ isc & \zeta s & i\zeta(1+i\zeta)(c^2+s^2) & c(1+i\zeta) \\ -\zeta s & isc(1+2\zeta^2) & c(1+i\zeta) & i\zeta(1+i\zeta) \end{bmatrix}\begin{bmatrix} c_e^+(N) \\ c_e^-(N') \\ c_h^-(N) \\ c_h^+(N') \end{bmatrix}, \tag{52}$$

where

$$D = (1+\zeta^2)c^2 + \zeta^2 s^2, \tag{53}$$

and in order to shorten the notations we have denoted

$$s \equiv \sinh(d_S/\xi), \quad c \equiv \cosh(d_S/\xi). \tag{54}$$

Referring to the notations introduced in Sec. II, we find from Eq. (52) that for an electronlike incident from the left

$$\mathcal{R} = \frac{\zeta^2(1+\zeta^2)(c^2+s^2)^2}{D^2}, \quad \mathcal{T} = \frac{c^2(1+\zeta^2)}{D^2},$$

$$\mathcal{R}_A = \frac{s^2c^2}{D^2}, \quad \mathcal{T}_A = \frac{\zeta^2 s^2}{D^2}, \tag{55}$$

while for an electronlike wave coming from the right the corresponding probabilities are

$$\mathcal{R}' = \frac{\zeta^2(1+\zeta^2)}{D^2}, \quad \mathcal{T}' = \frac{c^2(1+\zeta^2)}{D^2},$$

$$\mathcal{T}_A' = \frac{s^2c^2(1+2\zeta^2)^2}{D^2}, \quad \mathcal{T}_A' = \frac{\zeta^2 s^2}{D^2}. \tag{56}$$

It is easy to verify that the conditions for quasiparticle-number conservation, Eq. (5), are obeyed by the probabilities (55) and (56), since the scattering matrix is unitary; Eq. (6) for the charge conservation is not obeyed. Following Refs. 9 and 11, current conservation is now imposed on Eq. (7), leading to the determination of the chemical potential on the superconductor. This leads to a linear relation between I_L and the chemical-potential difference $\mu_L - \mu_R$, which is identified as the conductance. Denoting the latter by G_{sc}, one has

$$G_{\mathrm{sc}} = \frac{g_{LL}g_{RR} - g_{LR}g_{RL}}{g_{LL}+g_{RR}+g_{LR}+g_{RL}}, \tag{57}$$

where[9]

$$g_{ij} = \frac{2e^2}{h}(\delta_{ij} - |\mathcal{S}_{ij}^{ee}|^2 + |\mathcal{S}_{ij}^{he}|^2). \tag{58}$$

Here, i and j refer to the two sides of the junction, say left and right, and the superscripts *ee* or *he* refer to the particular process. Thus for example, the 11 element of the matrix in Eq. (52) is $\mathcal{S}_{LL}^{ee}$, while the 41 element is $\mathcal{S}_{RL}^{he}$.

We compare the outcome of Eq. (57) with the conductance found from the Kubo formula in Fig. 4. There, the conductances are plotted for four values of the interface transmission, an almost perfect one, $\mathcal{T}$=0.96, (the upmost pair of curves), $\mathcal{T}$=0.8, (the second pair of curves from above), $\mathcal{T}$=0.5 and $\mathcal{T}$=0.31, (the low-lying two pairs of curves). In each case, the result of Eq. (57) is the dashed line. There are three interesting features of this comparison. First, the conductance found from Eq. (57) is always *smaller* than the one found from the Kubo formula, Eq. (17). The difference between the two results decreases with increasing barrier (decreasing $\mathcal{T}$) and seems to vanish in the limit $\mathcal{T}\to 0$. Thus, while for a normal conductor the ring geometry and the simple two-terminal configuration produce identical results for the conductance, this is unfortunately no longer the case for the *NSN* junction (*NS* for the ring geometry). The second interesting feature concerns the slopes of the curves in Fig. 4, when the barrier transmission, $\mathcal{T}$ is varied. While the conductance computed from the Kubo formula shows a *crossover of the slope, from being positive at high values of* $\mathcal{T}$ to being negative at lower values, the slope of the conductance found from Eq. (57) seems to be *always negative*. A third important difference between the two approaches is that for large d_s/ξ and not-too-small $\mathcal{T}$, the Kubo result becomes larger than $2e^2/h$ (tending to $4e^2/h$ in the limit $d_s/\xi\to\infty$ *and* $\mathcal{T}\to 1$), while Eq. (57) actually tends to $G_{NS}=2e^2/h$ and *never* yields the doubling of G_{NS} due to the Andreev reflections. We blame these differences between the two approaches on the lack of conservation of charge in the BDG formulation. We believe that this deficiency is corrected by employing the Kubo formula for the ring geometry.

The difference between the two approaches becomes most marked in the limit of a nearly transparent barrier and a thick superconductor. This can be easily understood by noting that the addition of the two *NS* resistances is handled very differently by the two approaches. For the fully quantum case, adding two ideal conductances ($4e^2/h$ for the *NS* case) gives

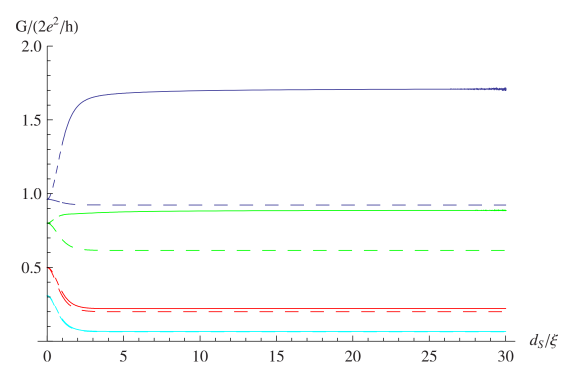

FIG. 4. (Color online) Comparison between the conductance Eq. (57) (dashed curves) and the conductance computed according to the Kubo formula, Eq. (17) (solid lines), as a function of d_S/ξ for four values of the interface transmission. The latter conductance is always larger than the former.

just one ideal conductance. On the other, the scattering formalism, in the limit $d_s/\xi \rightarrow \infty$ and $\mathcal{T} \rightarrow 1$ gives that both $\mathcal{T}$ ($\mathcal{T}'$) and $\mathcal{T}_A$ ($\mathcal{T}'_A$) vanish, and so does $\mathcal{R}$ ($\mathcal{R}'$), while $\mathcal{R}_A$ ($\mathcal{R}'_A$) tends to unity [see Eqs. (55) and (56)]. As a result, $g_{LL}=g_{RR}=4e^2/h$ and $g_{LR}=g_{RL}=0$, and Eq. (57) becomes exactly the classical addition of resistances, producing half the ideal quantum conductance of the pure *NS* junction. This is due to the fixing of the chemical potential on the *S* section to conserve the current, as in the classical treatment.

To recapitulate, we stress that unlike in the normal situation, where the two-terminal Landauer formula for the conductance was proven to agree with the Kubo low-frequency result,[18,19] here in the hybrid *SN* system this is no longer the case. For the closed ring, we conclude that the Kubo derivation, which follows from microscopic considerations valid in the linear-response regime without further assumptions, should capture the true behavior of the conductance.

In summary, we have shown that the presence of a superconducting segment in an otherwise normal system reduces the overall conductance once the barriers between the superconducting and the normal parts become high enough. Thus, the superconducting segments may push the system toward the localized insulating state.

From the appearance of the plots presented in Figs. 2 and 3, one may be tempted to say that the system experiences a metal-insulator quantum phase transition from a finite to a vanishing conductivity at large d_s/ξ when $\mathcal{T}$ decreases (which can be inferred to as "disorder increase"). We refrain here from making such a statement and defer the discussion of such a quantum phase transition in the thermodynamic limit for the composite *NS* system to future work.

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*oraentin@bgu.ac.il

†Also at Tel Aviv University, Tel Aviv 69978, Israel.

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